

# ENTROPY ALONG EXPANDING FOLIATIONS

JIAGANG YANG

**ABSTRACT.** The (measure-theoretical) entropy of a diffeomorphism along an expanding invariant foliation is the rate of complexity generated by the diffeomorphism along the leaves of the foliation. We prove that this number varies upper semi-continuously with the diffeomorphism ( $C^1$  topology), the invariant measure (weak\* topology) and the foliation itself in a suitable sense.

This has several important consequences. For one thing, it implies that the set of Gibbs  $u$ -states of  $C^{1+}$  partially hyperbolic diffeomorphism is an upper semi-continuous function of the map in the  $C^1$  topology. Another consequence is that the sets of partially hyperbolic diffeomorphisms with mostly contracting or mostly expanding center are  $C^1$  open.

## 1. INTRODUCTION

Partially hyperbolic diffeomorphisms were proposed by Brin, Pesin [11] and Pugh, Shub [28] independently at the early 1970's, as an extension of the class of Anosov diffeomorphisms [3, 4]. A diffeomorphism  $f$  is *partially hyperbolic* means that there exists a decomposition  $TM = E^s \oplus E^c \oplus E^u$  of the tangent bundle  $TM$  into three continuous invariant sub-bundles  $E_x^s$  and  $E_x^c$  and  $E_x^u$  such that  $Df|E^s$  is uniform contraction,  $Df|E^u$  is uniform expansion and  $Df|E^c$  lies in between them:

$$\frac{\|Df(x)v^s\|}{\|Df(x)v^c\|} \leq \frac{1}{2} \quad \text{and} \quad \frac{\|Df(x)v^c\|}{\|Df(x)v^u\|} \leq \frac{1}{2}$$

for any unit vectors  $v^s \in E^s$  and  $v^c \in E^c$  and  $v^u \in E^u$  and any  $x \in M$ .

Partially hyperbolic diffeomorphisms form an open subset of the space of  $C^r$  diffeomorphisms of  $M$ , for any  $r \geq 1$ . The *stable sub-bundle*  $E^s$  and the *unstable sub-bundle*  $E^u$  are uniquely integrable, that is, there are unique foliations  $\mathcal{F}^s$  and  $\mathcal{F}^u$  whose leaves are smooth immersed sub-manifolds of  $M$  tangent to  $E^s$  and  $E^u$ , respectively, at every point.

**1.1. Gibbs  $u$ -states.** Following Pesin and Sinai [27] and Bonatti and Viana [10] (see also [9, Chapter 11]), we call *Gibbs  $u$ -state* any invariant probability measure whose conditional probabilities (Rokhlin [31]) along strong unstable leaves are absolutely continuous with respect to the Lebesgue measure on the leaves. In fact, assuming the derivative  $Df$  is Hölder continuous, the Gibbs- $u$  state always exists, and the densities with respect to Lebesgue measures along unstable plaques are continuous. Moreover, the densities vary continuously with respect to the strong unstable leaves and the set of  $C^{1+\varepsilon}$  diffeomorphisms. As a consequence, the space of Gibbs  $u$ -states, denoted by  $\text{Gibb}^u(\cdot)$ , is compact relative to the weak-\* topology in the probability space, and varies upper semi-continuously with respect to the diffeomorphism in  $C^{1+\varepsilon}$  topology ([9, Remark 11.15]). In this article, we build a similar result in the  $C^1$  topology:

**Theorem A.**  *$\text{Gibb}^u(\cdot)$  varies upper semi-continuously among the  $C^{1+}$  partially hyperbolic diffeomorphisms in the  $C^1$  topology.*

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**1.2. Physical measures.** Let  $f : M \rightarrow M$  be a diffeomorphism on some compact Riemannian manifold  $M$ . An invariant probability  $\mu$  is a *physical measure* for  $f$  if the set of points  $z \in M$  for which

$$(1) \quad \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z)} \rightarrow \mu \quad (\text{in the weak}^* \text{ sense})$$

has positive volume. This set is denoted by  $B(\mu)$  and called the *basin* of  $\mu$ . A program for investigating the physical measures of partially hyperbolic diffeomorphisms was initiated by Alves, Bonatti, Viana in [5, 10], who proved existence and finiteness when  $f$  is either “mostly expanding” (asymptotic forward expansion) or “mostly contracting” (asymptotic forward contraction) along the center direction.

The set of Gibbs  $u$ -states plays important roles in the study of physical measures of partially hyperbolic diffeomorphisms. The partially hyperbolic diffeomorphisms with *mostly contracting center* are the  $C^{1+}$  partially hyperbolic diffeomorphisms whose Gibbs  $u$ -states have center Lyapunov exponents all negative, which were first studied in [10]. As a corollary of the semi continuation of the set of Gibbs  $u$ -states in the  $C^{1+\varepsilon}$  topology, the set of partially hyperbolic diffeomorphisms with mostly contracting center forms a  $C^{1+\varepsilon}$  open set (see [12, 1, 33, 16]).

The notation of partially hyperbolic diffeomorphisms with mostly expanding center was provided by Alves, Bonatti and Viana ([5]). More recently, Andersson and Vázquez proved in [2] that a partially hyperbolic diffeomorphism with every Gibbs  $u$ -state has center exponents all positive has mostly expanding center. They also proposed to the latter, somewhat stronger, property as the actual definition of mostly expanding center. We do that in the present paper. The reason is that we are going to prove that the set of diffeomorphisms satisfying this condition is open: that is not true for the original definition in [5], as observed in [2, Proposition A].

As a corollary of Theorem A, we obtain the  $C^1$  openness of the partially hyperbolic diffeomorphisms with mostly contracting center or with mostly expanding center.

**Theorem B.** *The sets of partially hyperbolic diffeomorphisms with mostly contracting center or mostly expanding center are  $C^1$  open, that is, every  $C^{1+}$  partially hyperbolic diffeomorphism with mostly contracting (resp. expanding) center admits a  $C^1$  open neighborhood, such that every  $C^{1+}$  diffeomorphism in this neighborhood has also mostly contracting (resp. expanding) center.*

The only known example of diffeomorphism with mostly expanding center (in the stronger sense we use in this paper, as explained above) is due to Mañé [24] (see [5] and [2, Section 6]). As an application of Theorem B, we provide a whole new class of example:

**Theorem C.** *Let  $f$  be a  $C^{1+}$  volume preserving partially hyperbolic diffeomorphism with one-dimensional center. Suppose the center exponent of the volume measure is positive and  $f$  is accessible, or the center exponent is negative, but the unstable foliation is minimal. Then  $f$  admits a  $C^1$  open neighborhood, such that every  $C^{1+}$  diffeomorphism in this neighborhood has mostly expanding center or mostly contracting center respectively, and it admits a unique physical measure, whose basin has full volume.*

Theorem C contains abundance of systems: By Avila [6],  $C^\infty$  volume preserving diffeomorphisms are  $C^1$  dense. And by Baraviera and Bonatti [7], the volume preserving partially hyperbolic diffeomorphisms with one-dimensional center and non-vanishing center exponent are  $C^1$  open and dense. Moreover, the subset of accessible systems is  $C^1$  open and  $C^k$  dense for any  $k \geq 1$  among all partially

hyperbolic diffeomorphisms with one-dimensional center direction, by Burns, Rodriguez Hertz, Rodriguez Hertz, Talitskaya and Ures [17, 14]; see also Theorem 1.5 in Nițică and Török [25]. The diffeomorphisms such that both the strong stable foliation and the strong unstable foliation are minimal are also quite common, which fill an open and dense subset of volume preserving partially hyperbolic diffeomorphisms with one-dimensional center and some fixed compact center leaf, this follows from a conservative version of the results of [8].

**1.3. Partial entropy for expanding foliations.** Let  $f$  be a  $C^1$  diffeomorphism, a foliation  $\mathcal{F}$  is  $f$ -*expanding* if it is invariant under  $f$  and the derivative  $Df$  restricted to the tangent bundle of  $\mathcal{F}$  is uniformly expanding. The *partial entropy* of an invariant probability measure along an expanding foliation is a value to measure the complexity of the measure generated on this foliation, which we will describe in Section 2.4 (see also [35]). Let  $\mu$  be any invariant measure of  $f$ , denote the partial entropy of  $\mu$  along the foliation  $\mathcal{F}$  by  $h_\mu(f, \mathcal{F})$ . Then Theorem A is implied by the upper semi-continuation of the partial metric entropy.

**Theorem D.** *Let  $f_n$  be a sequence of  $C^1$  diffeomorphisms which converge to  $f$  in the  $C^1$  topology, and  $\mu_n$  invariant measure of  $f_n$  which converge to an invariant measure  $\mu$  of  $f$  in the weak\* topology. Suppose  $\mathcal{F}_n$  is an expanding foliation of  $f_n$  for each  $n$  and  $\mathcal{F}_n \rightarrow \mathcal{F}$  in the sense of Definition 2.2, then*

$$\limsup h_{\mu_n}(f_n, \mathcal{F}_n^u) \leq h_\mu(f, \mathcal{F}_f^u).$$

The research on the regularity of entropy has a long history, one can find more references from [37, 26, 23, 34, 36]. Our proof of Theorem D is inspired by the dimension theory of invariant measures (see [38, 21, 22, 13]), and the inverse to the Pesin entropy formula ([19, 20, 21]).

**Outline of the work.** In section 2 we give the necessary material which will be used throughout the text. And in Section 3 we build a sequence of measurable partitions, which is used in Section 4 to prove Theorem D. Section 5 is devoted to the proofs of Theorems A and B. The proof of Theorem C is divided into Sections 6 and 7.

## 2. PRELIMINARY

Throughout this subsection, let  $f$  be a diffeomorphism of manifold  $M$ , and  $\mu$  an invariant probability measure of  $f$ .

**2.1. Volume preserving partially hyperbolic diffeomorphism.** We say a partially hyperbolic diffeomorphism is *accessible* if any two points can be joined by a piecewise smooth curve such that each leg is tangent to either  $E^u$  or  $E^s$  at every point.

Pugh, Shub conjectured in [29] that (essential) accessibility implies ergodicity, for a  $C^2$  partially hyperbolic, volume preserving diffeomorphism. In [30] they showed that this does hold under a few additional assumptions. The following result is a special case of a general result of Burns, Wilkinson [15]:

**Proposition 2.1.** *Every  $C^{1+\varepsilon}$  volume preserving, accessible partially hyperbolic diffeomorphism with one-dimensional center is ergodic.*

**2.2. Continuation of foliation.** In this subsection we explain the convergence between foliations that appeared in Theorem D.

Let  $\mathcal{F}$  be a foliation of  $M$  with dimension  $l$ , that is, every leaf is a  $l$ -dimensional smooth immersed submanifold. An  $\mathcal{F}$ -*foliation box* is some image  $B$  of a topological

embedding  $\Phi : D^{d-l} \times D^l \rightarrow M$  such that every plaque  $P_x = \Phi(\{x\} \times D^l)$  is contained in a leaf of  $\mathcal{F}$ , and every

$$\Phi(x, \cdot) : D^l \rightarrow M, y \mapsto \Phi(x, y)$$

is a  $C^1$  embedding depending continuously on  $x$  in the  $C^1$  topology. We write  $D = \Phi(D^{d-l} \times \{0\})$ , and denote this foliation box by  $(B, \Phi, D)$ .

Take a finite cover of  $M$  consists of  $\mathcal{F}$ -foliation boxes  $\{(B_i, \Phi_i, D_i)\}_{i=1}^k$ .

*Definition 2.2.* We say a sequence of  $l$ -dimensional foliations  $\mathcal{F}_n$  converge to  $\mathcal{F}$  if:

- for each  $n$ , there exists a finite cover of  $M$  by  $\mathcal{F}_n$ -foliation boxes

$$\{B_i^n, \Phi_i^n, D_i\}_{i=1}^k;$$

- for each  $1 \leq i \leq k$ , the topological embeddings  $\Phi_i^n : D^{d-l} \times D^l \rightarrow M$  converge uniformly to  $\Phi_i$  in the  $C^0$  topology;
- for every  $x \in D_i$  ( $1 \leq i \leq k$ ),  $\Phi^n(x, \cdot) : D^l \rightarrow M$  defined by

$$y \mapsto \Phi^n(x, y)$$

is a  $C^1$  embedding which converges to  $\Phi(x, \cdot)$  in the  $C^1$  topology as  $n \rightarrow \infty$ .

**2.3. Measurable partitions and mean conditional entropy.** Let  $\mathcal{B}$  be the Borel  $\sigma$ -algebra on  $M$ . In this subsection, we recall the properties of measurable partitions, more details see [31, 32].

*Definition 2.3.* A partition  $\xi$  of  $M$  is called *measurable* if there is a sequence of finite partitions  $\xi_n$ ,  $n \in \mathbb{N}$  such that:

- elements of  $\xi_n$  are measurable (up to  $\mu$ -measure 0);
- $\xi = \bigvee_n \xi_n$ , that is,  $\xi$  is the coarsest partition which refines  $\xi_n$  for each  $n$ .

For a partition  $\xi$  and  $x \in M$ , we denote by  $\xi(x)$  the element of  $\xi$  which contains  $x$ . For any measurable partition, we may define conditional measure on almost every element.

**Proposition 2.4.** *Let  $\xi$  be a measurable partition. Then there is a full  $\mu$ -measure subset  $\Gamma$  such that for every  $x \in \Gamma$ , there is a probability measure  $\mu_x^\xi$  defined on  $\xi(x)$  satisfying:*

- Let  $\mathcal{B}_\xi$  be the sub- $\sigma$ -algebra of  $\mathcal{B}$  which consist unions of elements of  $\xi$ , then for any measurable set  $A$ , the function  $x \rightarrow \mu_x^\xi(A)$  is  $\mathcal{B}_\xi$ -measurable.
- Moreover, we have

$$(2) \quad \mu(A) = \int \mu_x^\xi(A) d\mu(x).$$

*Remark 2.5.* Let  $\pi_\xi$  be the projection  $M \rightarrow M/\xi$ , and  $\mu_\xi$  be the projection of measure  $\mu$  onto  $M/\xi$  by the map  $\pi_\xi$ . Then equation (2) can be written as:

$$(3) \quad \mu(A) = \int \mu_B^\xi(A) d\mu_\xi(B)$$

where  $B$  denotes the element of  $\xi$  and  $\mu_B^\xi$  the conditional measure on  $B$ .

Let  $\xi$  be a measurable partition and  $C_1, C_2, \dots$  be the elements of  $\xi$  of positive measure. We define the *entropy of the partition* by

$$(4) \quad H_\mu(\xi) = \begin{cases} \sum_k \phi(\mu(C_k)), & \text{if } \mu(M \setminus \bigcup_k C_k) = 0 \\ \infty, & \text{if } \mu(M \setminus \bigcup_k C_k) > 0 \end{cases}$$

where  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$  is defined by  $\phi(x) = -x \log x$ .

If  $\xi$  and  $\eta$  are two measurable partitions, then for every element  $B$  of  $\eta$ ,  $\xi$  induces a partition  $\xi_B$  on  $B$ . We define the *mean conditional entropy of  $\xi$  respect to  $\eta$* , denoted by  $H_\mu(\xi | \eta)$ , as the following:

$$(5) \quad H_\mu(\xi | \eta) = \int_{M/\eta} H_{\mu_B^n}(\xi_B) d\mu_\eta(B).$$

**Definition 2.6.** For measurable partitions  $\{\zeta_n\}_{n=1}^\infty$  and  $\zeta$ , we write  $\zeta_n \nearrow \zeta$  if the following conditions are satisfied:

- $\zeta_1 < \zeta_2 < \dots$ ;
- $\bigvee_{n=1}^\infty \zeta_n = \zeta$ .

**Lemma 2.7.** [32, Subsection 5.11] Suppose  $\{\eta_n\}_{n=1}^\infty$ ,  $\eta$  and  $\xi$  are measurable partitions, such that  $\eta_n \nearrow \eta$  and  $H_\mu(\xi | \eta_1) < \infty$ , then

$$H_\mu(\xi | \eta_n) \searrow H_\mu(\xi | \eta).$$

**Definition 2.8.** Let  $\xi$  be a measurable partition, we put

$$h_\mu(f, \xi) = H_\mu(\xi | f\xi^+),$$

where  $\xi^+ = \bigvee_{n=0}^\infty f^n \xi$ .

**Remark 2.9.** A measurable partition  $\xi$  is said to be *increasing* if  $f\xi < \xi$ . For an increasing partition  $\xi$ ,

$$h_\mu(f, \xi) = H_\mu(\xi | f\xi).$$

**2.4. Expanding foliations.** Throughout this subsection,  $\mathcal{F}$  denotes an expanding foliation of  $f$ . We are going to give the precise definition of the partial metric entropy of  $\mu$  along the expanding foliation  $\mathcal{F}$ , which depends on a special class of measurable partitions:

**Definition 2.10.** We say a measurable partition  $\xi$  of  $M$  is  $\mu$ -subordinate to the  $\mathcal{F}$ -foliation if for  $\mu$ -a.e.  $x$ , we have

- (1)  $\xi(x) \subset \mathcal{F}(x)$  and  $\xi(x)$  has uniformly small diameter inside  $\mathcal{F}(x)$ ;
- (2)  $\xi(x)$  contains an open neighborhood of  $x$  inside the leaf  $\mathcal{F}(x)$ ;
- (3)  $\xi$  is an increasing partition, meaning that  $f\xi < \xi$ .

Ledrappier, Strelcyn [20] proved that the Pesin unstable lamination admits some  $\mu$ -subordinate measurable partition, the same proof can also be applied on general expanding foliations. Because in the following proof, we need a uniform construction of these partitions for a sequence of diffeomorphisms and measures, we providing the construction in Section 3.

The following result (for the subordinate partitions constructed as in Section 3) is contained in Lemma 3.1.2 of Ledrappier, Young [21]:

**Lemma 2.11.** Given any expanding foliation  $\mathcal{F}$ , we have  $h_\mu(f, \xi_1) = h_\mu(f, \xi_2)$  for any measurable partitions  $\xi_1$  and  $\xi_2$  that are  $\mu$ -subordinate to  $\mathcal{F}$ .

This allows us to give the following definition:

**Definition 2.12.** The *partial  $\mu$ -entropy*  $h_\mu(f, \mathcal{F})$  of the expanding foliation  $\mathcal{F}$  is defined by  $h_\mu(f, \xi)$  for any  $\mu$ -subordinate partition  $\xi$  constructed as in Section 3.

### 3. CONSTRUCTION OF SUBORDINATE MEASURABLE PARTITIONS

Let  $f_n$  be a sequence of diffeomorphisms which converge to  $f_0$  in the  $C^1$  topology, and  $\mathcal{F}_n$  an expanding foliation of  $f_n$  such that  $\mathcal{F}_n$  converge to  $\mathcal{F}_0$ . And  $\{(B_i^n, \Phi_i^n, D_i^n)\}_{i=1}^k$  and  $\{(B_i, \Phi_i, D_i)\}_{i=1}^k$  are the foliation boxes of  $\mathcal{F}_n$  and  $\mathcal{F}$  respectively as in the Definition 2.2. For simplicity, we assume that each plaque of every foliation box has diameter less than one.

The main aim of this section is to construct measurable partition  $\mu_n$ -subordinate to expanding foliation  $\mathcal{F}_n$  for each  $n$  in a uniform way, which is dealt in Lemma 3.2. The construction can be divided into two steps: The first step is to choose a finite partition  $\mathcal{A}$  of  $M$  such that

- every element of  $\mathcal{A}$  is contained in some foliation chart,
- the neighborhood of  $\partial\mathcal{A}$  has small measure for  $\mu$  and for every  $\mu_n$  where  $n \geq 1$  (see (9)).

Let  $\mathcal{A}^{\mathcal{F}}$  (resp.  $\mathcal{A}^{\mathcal{F}_n}$ ) be the partition such that every element is the intersection between an element of  $\mathcal{A}$  and a local  $\mathcal{F}$  (resp.  $\mathcal{F}_n$ ) plaque of the corresponding foliation box. Then the second step is to show that  $\vee_{i=0}^{\infty} f^i(\mathcal{A}^{\mathcal{F}})$  (resp.  $\vee_{i=0}^{\infty} f_n^i(\mathcal{A}^{\mathcal{F}_n})$ ) is subordinate to  $\mathcal{F}$  (resp.  $\mathcal{F}_n$ ).

Take  $r_0 \ll 1$  a Lebesgue number of the open covering  $\{B_i\}_{i=1}^k$ , that is, there is a function

$$(6) \quad I : M \rightarrow \{1, \dots, k\} \text{ such that } B_{r_0}(x) \subset B_{I(x)}.$$

When  $n$  is sufficiently large, by the definition of convergency of foliations, we have

$$(7) \quad B_{r_0}(x) \subset B_{I(x)}^n.$$

After removing a finite sequence, we assume (7) holds for every  $n \geq 1$ .

We need the following proposition whose proof we postpone to Appendix A

**Proposition 3.1.** *Let  $\{\nu_n\}_{n=0}^{\infty}$  be a sequence of probability measures on  $M$ . Then for any  $0 < \lambda < \lambda' < 1$  and  $R > 0$ , there is a finite partition  $\mathcal{A}$  of  $M$  and  $C_n > 0$  for every  $n \in \mathbb{N}$ , such that*

- the diameter of every element of  $\mathcal{A}$  is less than  $R$ ,
- $\nu_n(B_{\lambda^i}(\partial\mathcal{A})) \leq C_n(\lambda')^i$ , for every  $n, i \in \mathbb{N}$ , where  $B_r(\partial\mathcal{A})$  denotes the  $r$  neighborhood of  $\partial\mathcal{A}$ .

Take  $a > 1$  such that for any  $x \in M$  and  $n \geq 1$ ,

$$(8) \quad \|Df_n^{-1}|_{T_x\mathcal{F}_n(x)}\| < \frac{1}{a}.$$

Applying Proposition 3.1 for

- $\nu_n = \mu_n$  for any  $n \geq 0$ , where we write  $\mu_0 = \mu$ ;
- $R = r_0$ ;
- $\lambda = \frac{1}{a}$  and  $\frac{1}{a} < \lambda' < 1$ ,

we obtain a partition  $\mathcal{A}$  and  $C_n > 0$  ( $n \in \mathbb{N}$ ) such that  $\text{diam}(\mathcal{A}) < r_0$  and

$$(9) \quad \mu_n(B_{\lambda^i}(\partial\mathcal{A})) \leq C_n(\lambda')^i, \text{ for } n, i \geq 0.$$

Recall that every element of the partition  $\mathcal{A}^{\mathcal{F}}$  (resp.  $\mathcal{A}^{\mathcal{F}_n}$ ) is the intersection between  $\mathcal{A}$  and a local  $\mathcal{F}$  (resp.  $\mathcal{F}_n$ ) plaque in the corresponding foliation box.

**Lemma 3.2.**  *$\vee_{i=0}^{\infty} f^i(\mathcal{A}^{\mathcal{F}})$  is subordinate to  $\mathcal{F}$  and  $\vee_{i=0}^{\infty} f_n^i(\mathcal{A}^{\mathcal{F}_n})$  is subordinate to  $\mathcal{F}_n$  for every  $n > 0$ .*

*Proof of Lemma 3.2:* We only prove the first part of this lemma, the proof of the second part is similar.

Because  $\mu$  is  $f$  invariant, by (9), we have that

$$\sum_{j=1}^{\infty} \mu(f^j(B_{(\frac{1}{a})^j}(\partial\mathcal{A}))) = \sum_{j=1}^{\infty} \mu(B_{(\frac{1}{a})^j}(\partial\mathcal{A})) < \infty.$$

Hence, there is a  $\mu$  full measure subset  $Z$  and a function  $\mathcal{I} : Z \rightarrow \mathbb{N}$ , such that for every  $x \in Z$  and any  $j > \mathcal{I}(x)$ ,  $x \notin f^j(B_{(\frac{1}{a})^j}(\partial\mathcal{A}))$ , or equivalently,

$$(10) \quad f^{-j}(x) \notin B_{(\frac{1}{a})^j}(\partial\mathcal{A}).$$

Because  $\mu(\partial\mathcal{A}) = \mu(\bigcap B_{(\frac{1}{a})^j}(\partial\mathcal{A})) = 0$ , after removing a zero measure subset, we can assume that for every  $x \in Z$  and any  $j \in \mathbb{Z}$ ,  $f^j(x) \notin \partial\mathcal{A}$ . This hypothesis implies that for every  $m > 0$ ,  $\bigvee_{j=0}^m f^j(\mathcal{A}^{\mathcal{F}})(x)$  contains an open neighborhood of  $x$  inside the leaf  $\mathcal{F}(x)$ .

Then this lemma follows from the claim that  $\bigvee_{j=0}^m f^j(\mathcal{A}^{\mathcal{F}})(x) = \bigvee_{j=0}^{\mathcal{I}(x)} f^j(\mathcal{A}^{\mathcal{F}})(x)$  for any  $m \geq \mathcal{I}(x)$ , since this implies that  $\bigvee_{j=0}^{\infty} f^j(\mathcal{A}^{\mathcal{F}})(x) = \bigvee_{j=0}^{\mathcal{I}(x)} f^j(\mathcal{A}^{\mathcal{F}})(x)$ .

To prove this claim, we only need observe that every plaque of each foliation box  $(B_i, \Phi_i, D)$  has diameter bounded by 1. Suppose by contradiction that there is  $m \geq \mathcal{I}(x)$ , such that  $\bigvee_{j=0}^m f^j(\mathcal{A}^{\mathcal{F}})(x) \neq \bigvee_{j=0}^{m+1} f^j(\mathcal{A}^{\mathcal{F}})(x)$ . This implies that  $f^{m+1}(\partial\mathcal{A}) \cap \bigvee_{j=0}^m f^j(\mathcal{A}^{\mathcal{F}})(x) \neq \emptyset$ , i.e.,

$$d^{\mathcal{F}}(f^{m+1}(\partial\mathcal{A}), x) \leq 1,$$

where  $d^{\mathcal{F}}$  denotes the distance inside a leaf of the foliation  $\mathcal{F}$ . Then

$$d(f^{-(m+1)}(x), \partial\mathcal{A}) \leq d^{\mathcal{F}}(f^{-(m+1)}(x), \partial\mathcal{A}) \leq (\frac{1}{a})^{m+1},$$

which contradicts with (10), i.e.,  $f^{-(m+1)}(x) \notin B_{(\frac{1}{a})^{m+1}}(\partial\mathcal{A})$ .

We conclude the proof of this Lemma, and hence, complete the construction.  $\square$

From the construction, it is easy to show that:

**Lemma 3.3.**  $H_{\mu_n}(\mathcal{A}^{\mathcal{F}_n} | f_n(\mathcal{A}^{\mathcal{F}_n})) < \infty$ .

*Proof.* By the previous construction, the diameter of every element of  $\mathcal{A}$  is bounded by  $r_0$ , which is sufficiently small. Then every element  $B$  of the partition  $f_n(\mathcal{A}^{\mathcal{F}_n})$  is contained in a plaque of some foliation box. Moreover, the partition  $(\mathcal{A}^{\mathcal{F}_n})_B$  of  $B$  induced by  $\mathcal{A}^{\mathcal{F}_n}$  coincides to the partition of  $B$  induced by  $\mathcal{A}$ , which is uniform finite. Because the metric entropy of a partition is bounded by the logarithm of the number of its components, by (5), we have that

$$H_{\mu_n}(\mathcal{A}^{\mathcal{F}_n} | f_n(\mathcal{A}^{\mathcal{F}_n})) = \int_{M/f_n(\mathcal{A}^{\mathcal{F}_n})} H_{(\mu_n)_{f_n(\mathcal{A}^{\mathcal{F}_n})}}((\mathcal{A}^{\mathcal{F}_n})_B) d(\mu_n)_{f_n(\mathcal{A}^{\mathcal{F}_n})}(B)$$

is bounded by the logarithm of the number of the components of  $\mathcal{A}$ . The proof is finished.  $\square$

#### 4. APPROACH OF PARTIAL ENTROPY

In this section we give the proof of Theorem D.

For simplicity, we denote by  $f_0 = f$ ,  $\mu_0 = \mu$ ,  $\mathcal{F}_0 = \mathcal{F}$ , and the foliation boxes

$$\{(B_i^0, \Phi_i^0, D_i)\}_{i=1}^k = \{(B_i, \Phi_i, D_i)\}_{i=1}^k.$$

Let  $\{(B_i^n, \Phi_i^n, D_i)\}_{i=1}^k$  be the foliation boxes of  $\mathcal{F}_n$  as in the Definition 2.2, and  $\mathcal{A}$  and  $\mathcal{A}^{\mathcal{F}_n}$  be the partitions constructed in the previous section.

**4.1. First approach:** In the subsection, we use the partition  $\mathcal{A}^{\mathcal{F}_n}$  to calculate the partial metric entropy of  $\mu_n$  along the expanding foliation  $\mathcal{F}_n$ .

**Proposition 4.1.** *For every  $n \geq 0$ , we have*

$$h_{\mu_n}(f_n, \mathcal{F}_n) = \lim_{m \rightarrow \infty} \frac{1}{m} H_{\mu_n}(\bigvee_{j=1}^m f_n^{-j}(\mathcal{A}^{\mathcal{F}_n}) | \mathcal{A}^{\mathcal{F}_n}) = \inf_{m \rightarrow \infty} \frac{1}{m} H_{\mu_n}(\bigvee_{j=1}^m f_n^{-j}(\mathcal{A}^{\mathcal{F}_n}) | \mathcal{A}^{\mathcal{F}_n}).$$

*Proof.* By the property of conditional entropy ([32, Subsection 5.9]),

$$\begin{aligned} H_{\mu_n}(\bigvee_{j=1}^m f_n^{-j}(\mathcal{A}^{\mathcal{F}_n}) | \mathcal{A}^{\mathcal{F}_n}) &= H_{\mu_n}(f_n^{-1}(\mathcal{A}^{\mathcal{F}_n}) | \mathcal{A}^{\mathcal{F}_n}) + \dots \\ &\quad + H_{\mu_n}(f_n^{-m}(\mathcal{A}^{\mathcal{F}_n}) | \bigvee_{j=0}^{m-1} f_n^{-j}(\mathcal{A}^{\mathcal{F}_n})). \end{aligned}$$

Because  $\mu_n$  is  $f_n$  invariant, it follows that

$$\begin{aligned}
 H_{\mu_n}(\vee_{j=1}^m f_n^{-j}(\mathcal{A}^{\mathcal{F}_n}) | \mathcal{A}^{\mathcal{F}_n}) &= H_{\mu_n}(\mathcal{A}^{\mathcal{F}_n} | f_n(\mathcal{A}^{\mathcal{F}_n})) + \dots \\
 &\quad + H_{\mu_n}(\mathcal{A}^{\mathcal{F}_n} | \vee_{j=0}^{m-1} f_n^{m-j}(\mathcal{A}^{\mathcal{F}_n})) \\
 &= \sum_{j=1}^m H_{\mu_n}(\mathcal{A}^{\mathcal{F}_n} | \vee_{j=1}^i f_n^j(\mathcal{A}^{\mathcal{F}_n})).
 \end{aligned}
 \tag{11}$$

Since  $\vee_{j=1}^i f_n^j(\mathcal{A}^{\mathcal{F}_n})$  is an increasing sequence, by Lemmas 2.7 and 3.3, we have

$$H_{\mu_n}(\mathcal{A}^{\mathcal{F}_n} | \vee_{j=1}^i f_n^j(\mathcal{A}^{\mathcal{F}_n})) \searrow H_{\mu_n}(\mathcal{A}^{\mathcal{F}_n} | \vee_{j=1}^{\infty} f_n^j(\mathcal{A}^{\mathcal{F}_n})) = h_{\mu_n}(f_n, \mathcal{F}_n).$$

Then by (11):

$$\lim_{m \rightarrow \infty} \frac{1}{m} H_{\mu_n}(\vee_{j=1}^m f_n^{-j}(\mathcal{A}^{\mathcal{F}_n}) | \mathcal{A}^{\mathcal{F}_n}) = \inf_{m \rightarrow \infty} \frac{1}{m} H_{\mu_n}(\vee_{j=1}^m f_n^{-j}(\mathcal{A}^{\mathcal{F}_n}) | \mathcal{A}^{\mathcal{F}_n}) = h_{\mu_n}(f_n, \mathcal{F}_n).$$

□

**4.2. Second approach.** In this subsection, we use the conditional entropy between two finite partitions to approach the partial  $\mu_n$ -entropy of the expanding foliation  $\mathcal{F}_n$ . We begin by the following easy observation, where  $\mathcal{A}_n^m = \vee_{j=0}^m f_n^{-j} \mathcal{A}$ .

**Lemma 4.2.** *For every  $m > 0$ ,  $1 \leq i \leq k$  and  $x \in B_i$ ,*

$$\vee_{j=0}^m f_n^{-j}(\mathcal{A}^{\mathcal{F}_n})(x) = \mathcal{A}_n^m(x) \cap \mathcal{A}^{\mathcal{F}_n}(x).$$

*Proof.* Denote by  $\mathcal{F}_{n,\text{loc}}(x)$  the local plaque of foliation  $\mathcal{F}_n(x)$  which contains  $x$ .

Suppose by contradiction that there is  $y \in \mathcal{A}_n^m(x)$  such that  $y \in \mathcal{F}_{n,\text{loc}}(x)$  but  $\vee_{j=0}^m f_n^{-j}(\mathcal{A}^{\mathcal{F}_n})(y) \neq \vee_{j=0}^m f_n^{-j}(\mathcal{A}^{\mathcal{F}_n})(x)$ . Let  $0 < k \leq m$  be the number such that

- $y_j = f^j(y)$  and  $x_j = f^j(x)$  belong to the same elements of  $\mathcal{A}^{\mathcal{F}_n}$  for every  $0 \leq j < k$ ;
- $y_k$  and  $x_k$  belong to different elements of  $\mathcal{A}^{\mathcal{F}_n}$ .

Because  $\mathcal{A}$  has small diameter,  $\mathcal{A}^{\mathcal{F}_n}(y_{k-1}) = \mathcal{A}^{\mathcal{F}_n}(x_{k-1})$  also has small diameter, which implies that  $f_n(\mathcal{A}^{\mathcal{F}_n}(y_{k-1}))$  is contained in  $\mathcal{F}_{n,\text{loc}}(x_k)$ . Then by the definition of  $\mathcal{A}^{\mathcal{F}_n}$ :

$$y_k \in \mathcal{F}_{n,\text{loc}}(x_k) \cap \mathcal{A}(x_k) = \mathcal{A}^{\mathcal{F}_n}(x_k),$$

a contradiction to the assumption. □

**4.2.1. New partitions:** Let  $\mathcal{C}_{i,1} \leq \mathcal{C}_{i,2} \leq \dots$  be a sequence of finite partitions on  $D^i$  such that

- (A)  $\text{diam}(\mathcal{C}_{i,t}) \rightarrow 0$ ;
- (B) for any  $i, t \geq 0$  and any element  $C$  of  $\mathcal{C}_{i,t}$ :  $\mu_n(\cup_{x \in \partial C} \mathcal{A}^{\mathcal{F}_n}(x)) = 0$ .

For every  $i, t \geq 0$ , the partition  $\mathcal{C}_{i,t}$  induces a partition  $\tilde{\mathcal{C}}_{n,i,t}$  on the foliation box  $B_i^n$ :

$$\tilde{\mathcal{C}}_{n,i,t} = \{\cup_{x \in C} \mathcal{A}^{\mathcal{F}_n}(x); C \text{ is an element of } \mathcal{C}_{i,t}\}.$$

**Remark 4.3.** Because  $\text{diam}(\mathcal{C}_{i,t}) \rightarrow 0$ , for any  $x \in B_n^i$ ,  $\tilde{\mathcal{C}}_{n,i,t}(x) \rightarrow \mathcal{A}_{\mathcal{F}_n}(x)$ .

For an element  $P$  of  $\mathcal{A}_n^m$ , suppose that  $I|_P = i$  (see (6) on the definition of the function  $I(x)$ ), which implies that  $P \subset B_i^n$ . Then  $\tilde{\mathcal{C}}_{n,i,t}$  induces on  $P$  a partition  $P_t$ :  $\{P \cap \tilde{C}; \tilde{C} \text{ is an element of } \tilde{\mathcal{C}}_{n,i,t}\}$ . And for every  $m, n, t \geq 0$ ,

$$\mathcal{A}_{n,t}^m = \{P_t; \text{ where } P \in \mathcal{A}_n^m\}$$

is a new partition of the ambient manifold  $M$ . In the following we identify some properties for the new partition, which are important for the further proof.

**Lemma 4.4.** *For any  $n, m \geq 0$ :*



- (i)  $\mathcal{A}_{n,t}^m \nearrow_{t \rightarrow \infty} \bigvee_{j=0}^m f_n^{-j} \mathcal{A}^{\mathcal{F}_n}$ ;
- (ii)  $\mathcal{A}_n^m < \mathcal{A}_{n,t}^m < \bigvee_{j=0}^m f_n^{-j} \mathcal{A}^{\mathcal{F}_n}$ ;
- (iii)  $\mu_n(\partial \mathcal{A}_{n,t}^m) = 0$ .

*Proof.* From the construction of the partition  $\mathcal{A}_{n,t}^m$ , (i) and (ii) follow immediately. Moreover,

$$\partial \mathcal{A}_{n,t}^m \subset \partial \mathcal{A}_n^m \bigcup \bigcup_{i=1}^k \bigcup_{C \in \mathcal{C}_{i,t}} \bigcup_{x \in \partial C} \mathcal{A}^{\mathcal{F}_n}(x).$$

By the assumption (B) above,  $\mu_n(\bigcup_{x \in \partial C} \mathcal{A}^{\mathcal{F}_n}(x)) = 0$ . Note also that by (9),  $\mu_n(\partial \mathcal{A}_n^m) = 0$ . The proof is complete.  $\square$

The following proposition is the key for the approach:

**Proposition 4.5.**  $H_{\mu_n}(\mathcal{A}_{n,t}^m \mid \mathcal{A}_{n,t}^0) \searrow_{t \rightarrow \infty} H_{\mu_n}(\bigvee_{j=0}^m f_n^{-j} \mathcal{A}^{\mathcal{F}_n} \mid \mathcal{A}^{\mathcal{F}_n})$ .

*Proof.* We first claim that for  $t_1 < t_2$ ,

$$\mathcal{A}_{n,t_1}^m \cap \mathcal{A}_{n,t_2}^0 = \mathcal{A}_{n,t_2}^m \cap \mathcal{A}_{n,t_2}^0.$$

Observe that every component  $B \in \mathcal{A}_{n,t_1}^m \cap \mathcal{A}_{n,t_2}^0$  is the intersection of components:  $C \in \mathcal{A}_n^m$ ,  $D \in \tilde{\mathcal{C}}_{n,i,t_1}$ ,  $E \in \mathcal{A}_n^0$  and  $F \in \tilde{\mathcal{C}}_{n,i,t_2}$ . Because  $\tilde{\mathcal{C}}_{n,i,t_2}$  is finer than  $\tilde{\mathcal{C}}_{n,i,t_1}$ , we have  $D \supset F$ , which implies that

$$B = C \cap D \cap E \cap F = (C \cap F) \cap (E \cap F)$$

is a component of the partition  $\mathcal{A}_{n,t_2}^m \cap \mathcal{A}_{n,t_2}^0$ , as claimed.

Fix  $t_1 = 1$ , the above claim implies in particular that

$$\begin{aligned} H_{\mu_n}(\mathcal{A}_{n,t_2}^m \mid \mathcal{A}_{n,t_2}^0) &= H_{\mu_n}(\mathcal{A}_{n,t_2}^m \vee \mathcal{A}_{n,t_2}^0 \mid \mathcal{A}_{n,t_2}^0) \\ (13) \quad &= H_{\mu_n}(\mathcal{A}_{n,1}^m \vee \mathcal{A}_{n,t_2}^0 \mid \mathcal{A}_{n,t_2}^0) \\ &= H_{\mu_n}(\mathcal{A}_{n,1}^m \mid \mathcal{A}_{n,t_2}^0). \end{aligned}$$

Applying Lemma 4.4 (i) on  $m = 0$ ,  $\mathcal{A}_{n,t}^0 \nearrow \mathcal{A}^{\mathcal{F}_n}$ . Because both partitions  $\mathcal{A}_{n,1}^m$  and  $\mathcal{A}_{n,1}^0$  are finite,  $H_{\mu_n}(\mathcal{A}_{n,1}^m \mid \mathcal{A}_{n,1}^0) < \infty$ . By (13) and Lemma 2.7, we have

$$\begin{aligned} H_{\mu_n}(\mathcal{A}_{n,t}^m \mid \mathcal{A}_{n,t}^0) &= H_{\mu_n}(\mathcal{A}_{n,1}^m \mid \mathcal{A}_{n,t}^0) \\ &\searrow H_{\mu_n}(\mathcal{A}_{n,1}^m \mid \mathcal{A}^{\mathcal{F}_n}) \\ &= H_{\mu_n}(\mathcal{A}_{n,1}^m \vee \mathcal{A}^{\mathcal{F}_n} \mid \mathcal{A}^{\mathcal{F}_n}). \end{aligned}$$

Then the proof follows from the claim that  $\mathcal{A}_{n,t}^m \vee \mathcal{A}^{\mathcal{F}_n} = \bigvee_{j=0}^m f_n^{-j} \mathcal{A}^{\mathcal{F}_n}$ .

It remains to prove the claim, which is a corollary of Lemma 4.4 (ii) by taking  $t = 1$ : On one hand,  $\mathcal{A}_{n,1}^m < \bigvee_{j=0}^m f_n^{-j} \mathcal{A}^{\mathcal{F}_n}$ , which implies that

$$(14) \quad \mathcal{A}_{n,1}^m \vee \mathcal{A}^{\mathcal{F}_n} < \bigvee_{j=0}^m f_n^{-j} \mathcal{A}^{\mathcal{F}_n} \vee \mathcal{A}^{\mathcal{F}_n} = \bigvee_{j=0}^m f_n^{-j} \mathcal{A}^{\mathcal{F}_n}.$$

On the other hand,  $\mathcal{A}_n^m < \mathcal{A}_{n,1}^m$ . Then

$$\mathcal{A}_n^m \vee \mathcal{A}^{\mathcal{F}_n} < \mathcal{A}_{n,1}^m \vee \mathcal{A}^{\mathcal{F}_n}.$$

By Lemma 4.2,

$$(15) \quad \bigvee_{j=0}^m f_n^{-j} \mathcal{A}^{\mathcal{F}_n} < \mathcal{A}_{n,1}^m \vee \mathcal{A}^{\mathcal{F}_n}.$$

We conclude the proof of the claim by (14) and (15).  $\square$

**Corollary 4.6.**  $H_{\mu_n}(\mathcal{A}_{n,t}^m \mid \mathcal{A}_{n,t}^0) > h_{\mu_n}(f_n, \mathcal{F}_n)$ .

*Proof.* This is a consequence of Propositions 4.1 and 4.5.  $\square$

**4.3. Proof of Theorem D.** By Proposition 4.1, for any  $\varepsilon > 0$ , there is  $m_0$  sufficiently large, such that

$$(16) \quad \frac{1}{m_0} H_\mu(\vee_{j=0}^{m_0} f^{-j} \mathcal{A}^\mathcal{F} | \mathcal{A}^\mathcal{F}) - \frac{\varepsilon}{3} \leq h_\mu(f, \varepsilon)$$

By Proposition 4.5, we may further take  $t_0 > 0$  large, such that

$$(17) \quad H_\mu(\mathcal{A}_{0,t_0}^{m_0} | \mathcal{A}_{0,t_0}^0) - \frac{\varepsilon}{3} \leq H_\mu(\vee_{j=0}^{m_0} f^{-j} \mathcal{A}^\mathcal{F} | \mathcal{A}^\mathcal{F})$$

Because  $f_n$  converge to  $f$  in the  $C^1$  topology, and the foliations  $\mathcal{F}_n$  converge to  $\mathcal{F}$  (see Definition 2.2), each component of  $P_0 \in \mathcal{A}_{0,t_0}^{m_0}$  is converged by the corresponding component  $P_n$  of the partition  $\mathcal{A}_{n,t_0}^{m_0}$  in the Hausdorff topology. Note that  $\mu_n$  converge to  $\mu$  in the weak\* topology, and by Lemma 4.4 (iii),  $\mu(\partial P_0) = 0$ , hence,

$$\lim_{n \rightarrow \infty} \mu_n(P_n) = \mu(P_0).$$

Because  $\mathcal{A}_{0,t_0}^{m_0}$  is a finite partition, we have

$$(18) \quad \lim_{n \rightarrow \infty} H_{\mu_n}(\mathcal{A}_{n,t_0}^{m_0} | \mathcal{A}_{n,t_0}^0) = H_\mu(\mathcal{A}_{0,t_0}^{m_0} | \mathcal{A}_{0,t_0}^0).$$

Then there is  $n_0$  large enough, such that for any  $n \geq n_0$ ,

$$H_{\mu_n}(\mathcal{A}_{n,t_0}^{m_0} | \mathcal{A}_{n,t_0}^0) - \frac{\varepsilon}{3} \leq H_\mu(\mathcal{A}_{0,t_0}^{m_0} | \mathcal{A}_{0,t_0}^0).$$

Combining (16) and (17), for any  $n \geq n_0$ , one has

$$H_{\mu_n}(\mathcal{A}_{n,t_0}^{m_0} | \mathcal{A}_{n,t_0}^0) - \varepsilon \leq h_\mu(f, \mathcal{F}).$$

By Corollary 4.6, for any  $n \geq n_0$ ,

$$h_{\mu_n}(f_n, \mathcal{F}_n) - \varepsilon \leq h_\mu(f, \mathcal{F}).$$

Because  $\varepsilon$  can be taken arbitrarily small, we conclude the proof of Theorem D.

## 5. GIBBS $u$ -STATES OF PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

Let  $f$  be a  $C^{1+}$  partially hyperbolic diffeomorphism with invariant splitting on the tangent bundle:  $T_x M = E_x^s \oplus E_x^c \oplus E_x^u$ . Denote by  $\mathcal{F}^u$  the unstable foliation of  $f$  which is tangent to the unstable bundle  $E^u$ , and write  $\text{Jac}^u(x) = \det Df|_{E_x^u}$ .

**5.1. Preliminaries for Gibbs  $u$ -states.** Denote by  $\text{Gibb}^u(f)$  the set of Gibbs  $u$ -states of  $f$ . The proofs for the following basic properties of Gibbs  $u$ -states can be found in Bonatti, Díaz and Viana [9, Subsection 11.2]:

- Proposition 5.1.** (1)  *$\text{Gibb}^u(f)$  is non-empty, weak\* compact and convex. Ergodic components of Gibbs  $u$ -states are Gibbs  $u$ -states.*  
 (2) *The support of every Gibbs  $u$ -state is  $\mathcal{F}^u$ -saturated, that is, it consists of entire strong unstable leaves.*  
 (3) *For Lebesgue almost every point  $x$  in any disk inside some strong unstable leaf, every accumulation point of  $\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$  is a Gibbs  $u$ -state.*  
 (4) *Every physical measure of  $f$  is a Gibbs  $u$ -state and, conversely, every ergodic  $u$ -state whose center Lyapunov exponents are negative is a physical measure.*

In the following, we show an upper bound for the partial entropy along the expanding foliation  $\mathcal{F}^u$ , which is similar to the Ruelle inequality.

**Proposition 5.2.** *Let  $\mu$  be an invariant probability measure of  $f$ , then*

$$h_\mu(f, \mathcal{F}^u) \leq \int \log \text{Jac}^u(x) d\mu(x).$$

Moreover,

$$(19) \quad h_\mu(f, \mathcal{F}^u) = \int \log \text{Jac}^u(x) d\mu(x).$$

if and only if  $\mu$  is a Gibbs  $u$ -state of  $f$ .

*Proof.* By [22, Theorem  $C'$ ], when  $\mu$  is ergodic and  $f$  is  $C^2$ , then

$$h_\mu(f, \mathcal{F}) \leq \int \log \text{Jac}^u(x) d\mu(x).$$

It was pointed out by [13, p. 761] that the same inequality goes well for  $C^{1+}$  diffeomorphism. When  $\mu$  is not ergodic, one only need observe that the metric entropy function is an affine function. Then by the ergodic decomposition, we conclude the proof of the first part of the proposition.

The second part was stated in [19, Theorem 3.4]. □

The following equality was built in [22, Proposition 5.1], when  $\mu$  is ergodic and  $f$  is  $C^2$ . As explained above, which also holds for general situation:

**Proposition 5.3.** *Let  $\mu$  be a probability measure of  $f$  such that the center exponents of  $\mu$  are all vanishing, then*

$$h_\mu(f, \mathcal{F}) = h_\mu(f).$$

**5.2. Diffeomorphisms with mostly expanding/contracting center.** In this Subsection we state equivalent definition for diffeomorphisms with mostly contracting (resp. expanding) center direction which was proved in [1] (resp. [2]). For completeness, we provide the proofs here.

**Proposition 5.4.** *[[1]] A diffeomorphism  $f$  has mostly contracting center if and only if there is  $N \in \mathbb{N}$  and  $a > 0$  such that for any  $\mu \in \text{Gibb}^u(f)$ ,*

$$(20) \quad \int \log \|Df^N|_{E^{cs}(x)}\| d\mu(x) < -a.$$

**Proposition 5.5.** *[[2]] A diffeomorphism  $f$  has mostly expanding center if and only if there is  $N \in \mathbb{N}$  and  $a > 0$  such that for any  $\mu \in \text{Gibb}^u(f)$ ,*

$$(21) \quad \int \log \|Df^{-N}|_{E^{cs}(x)}\| d\mu(x) < -a.$$

*Proof of Proposition 5.4:* Suppose  $f$  has mostly contracting center. Then for every Gibbs  $u$ -state  $\mu$  of  $f$ , the integration of the largest center exponent of  $\mu$  is negative, that is,

$$\lim_n \int \frac{1}{n} \log \|Df^n|_{E^{cs}(x)}\| d\mu < 0.$$

There exists  $N_\mu > 1$  and  $a_\mu > 0$  such that

$$(22) \quad \int \log \|Df^{N_\mu}|_{E^{cs}}\| d\mu < -N_\mu a_\mu.$$

We may take a neighborhood  $\mathcal{V}_\mu$  of  $\mu$  inside the probability measure space of  $M$ , such that, (22) holds for any probability measure  $\nu \in \mathcal{V}_\mu$ :

$$(23) \quad \int \log \|Df^{N_\mu}|_{E^{cs}}\| d\nu \leq -N_\mu a_\mu.$$

Because the space of Gibbs  $u$ -states of  $f$  is compact (see Proposition 5.1 (1)), there is a finite open covering  $\{\mathcal{V}_{\mu_j}\}_{j=1}^k$  of  $\text{Gibb}^u(f)$ . For simplicity, we write

$N_j = N_{\mu_j}$  and  $a_j = a_{\mu_j}$ . Let  $N = \prod_{j=1}^k N_j$  and  $a = \min\{a_1, \dots, a_k\}$ . For any Gibbs  $u$ -state  $\mu$  of  $f$ , there is  $1 \leq j_0 \leq k$  such that  $\mu \in \mathcal{V}_{j_0}$ . Because

$$Df^N|_{E^{cs}(x)} = Df^{N_{j_0}}|_{E^{cs}(f^{N-N_{j_0}}(x))} \circ \dots \circ Df^{N_{j_0}}|_{E^{cs}(x)},$$

by (23),

$$(24) \quad \begin{aligned} \int \log \|Df^N|_{E^{cs}}\| d\mu &\leq \frac{N}{N_{j_0}} \int \log \|Df^{N_{j_0}}|_{E^{cs}}\| d\mu \\ &\leq -\frac{N}{N_{j_0}} a_{j_0} \leq -a. \end{aligned}$$

On the contrary, now we assume that (20) holds for any Gibbs  $u$ -state  $\mu$  of  $f$ , which implies that the center exponents of any ergodic Gibbs  $u$ -state of  $f$  are all negative. Because the ergodic components of every Gibbs  $u$ -state are still Gibbs  $u$ -states (see Proposition 5.1 (1)), we then conclude that for any Gibbs  $u$ -state  $\mu$  of  $f$ , the center exponents of  $\mu$  almost every point are all negative. The proof of Proposition 5.4 is complete.  $\square$

The proof of Proposition 5.5 is quite similar: we only need replace the diffeomorphism  $f$  in the proof above by its inverse  $f^{-1}$ . We will not detail the proof here.

**5.3. Proof of Theorem A.** Instead of proving Theorem A, we will prove the following equivalent proposition:

**Proposition 5.6.** *Let  $\{f_n\}_1^\infty$  be a sequence of  $C^{1+}$  partially hyperbolic diffeomorphisms, and  $\mu_n$  Gibbs- $u$  state of  $f_n$ . Suppose  $f_n$  converge to a  $C^{1+}$  diffeomorphism  $f$  in the  $C^1$  topology, and  $\mu_n$  converge to a probability measure  $\mu$  in the weak- $*$  topology, then  $\mu$  is a Gibbs- $u$  state of  $f$ .*

Denote by  $\text{Jac}_n(x) = \det Df_n|_{E_x^u}$ , and  $\mathcal{F}_n^u$  the unstable foliation of  $f_n$ . Because  $\mu_n$  is a Gibbs  $u$ -state of  $f_n$  for each  $n$ , by Proposition 5.2,

$$h_{\mu_n}(f_n, \mathcal{F}_n^u) = \int \log \text{Jac}_n(x) d\mu_n(x).$$

By the unstable manifold theorem of [18], the foliations  $\mathcal{F}_n^u$  converge to  $\mathcal{F}^u$  as in Definition 2.2. Then, as a corollary of Theorem D,

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(f_n, \mathcal{F}_n^u) \leq h_\mu(f, \mathcal{F}^u).$$

Note that  $\{\text{Jac}_n(\cdot)\}$  are continuous functions which are bounded from below by zero and converge uniformly to  $\text{Jac}(\cdot)$ , we have

$$\lim_{n \rightarrow \infty} \int \log \text{Jac}_n(x) d\mu_n(x) = \int \log \text{Jac}(x) d\mu(x).$$

Therefor,  $h_\mu(f, \mathcal{F}^u) \geq \int \log \text{Jac}(x) d\mu(x)$ . But by the first part of Proposition 5.2,

$$h_\mu(f, \mathcal{F}^u) \leq \int \log \text{Jac}(x) d\mu(x).$$

Hence, we have the equality

$$h_\mu(f, \mathcal{F}^u) = \int \log \text{Jac}(x) d\mu(x).$$

By the second part of Proposition 5.2,  $\mu$  is a Gibbs  $u$ -state of  $f$ . The proof is complete.

**5.4. Proof of Theorem B.** We begin by showing that the set of diffeomorphisms with mostly contracting center is  $C^1$  open:

Let  $f$  be a  $C^{1+}$  partially hyperbolic diffeomorphism with mostly contracting center. By Proposition 5.4, there is  $N > 1$  and  $a > 0$  such that for any Gibbs  $u$ -state  $\mu$  of  $f$ ,

$$(25) \quad \int \log \|Df^N|_{E^{cs}}\| d\mu < -a.$$

By Theorem A, for any small neighborhood  $\mathcal{V}$  of  $\text{Gibb}^u(f)$  in the space of probability measures of  $M$ , there is a  $C^1$  neighborhood  $\mathcal{U}$  of  $f$ , such that for any  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$ ,  $\text{Gibb}^u(g) \subset \mathcal{V}$ . Because the bundle  $E^{cs}$  varies continuously with respect to the diffeomorphisms in the  $C^1$  topology, we may take the neighborhood  $\mathcal{V}$  of  $\text{Gibb}^u(f)$  sufficiently small, and  $\mathcal{U}$  the neighborhood of  $f$  small enough, such that for any  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$ ,

$$\int \log \|Dg^N|_{E^{cs}}\| d\mu < -a \text{ for any Gibbs } u\text{-state } \mu \text{ of } g.$$

Then by Proposition 5.4, every  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$  has mostly contracting center direction.

Replacing the diffeomorphism  $f$  in the above proof by  $f^{-1}$ , and making use of Proposition 5.5 instead of Proposition 5.4, one may show that the set of partially hyperbolic diffeomorphisms with mostly expanding center is  $C^1$  open.

## 6. DIFFEOMORPHISMS WITH MOSTLY EXPANDING CENTER

We prove the first part of Theorem C in this section. The proof is divided into two parts: In Subsection 6.2 we show the existence of physical measures for diffeomorphisms with mostly expanding center, moreover, these physical measures admit uniform size of basins in a robust way. In Subsection 6.3 we prove Theorem C in the case of diffeomorphisms with positive center exponent.

Throughout this section, let  $f : M \rightarrow M$  be a  $C^{1+}$  partially hyperbolic diffeomorphism.

**6.1. Gibbs  $cu$ -states.** In this subsection we recall the main result of [5] on the existence of physical measures for the diffeomorphisms which are no-uniformly expanding along the center-unstable direction (see the definition below). We will outline the argument and explain that these physical measures indeed have uniform size of basins.

*Definition 6.1.* For  $a > 0$ , we say  $f$  is a *no-uniformly expanding along the center-unstable direction*, if

$$(26) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log \|Df^{-1}|_{E^{cu}(f^j(x))}\| < -a < 0,$$

for Lebesgue almost every  $x \in M$ .

**Theorem 6.2** ([5] Theorem A). *Assuming that  $f$  is no-uniformly expanding along the center-unstable direction. Then  $f$  has finitely many physical measures, whose basins cover a full volume subset of the ambient manifold.*

In the blow, we give more precise description of these physical measures.

### 6.1.1. Hyperbolic time.

**Definition 6.3.** Given  $a > 0$ , we say  $n$  is an  $a$ -hyperbolic time for a point  $x$  if

$$\frac{1}{k} \sum_{j=n-k+1}^n \log \|Df^{-1}|_{E^{cu}(f^j(x))}\| \leq -a \text{ for any } 0 < j \leq n.$$

Let  $D$  be any  $C^1$  disk, we use  $d_D(\cdot, \cdot)$  denotes the distance between two points in the disk.

**Lemma 6.4** ([5] Lemma 2.7). *For any  $a > 0$ , there is  $\delta_1 > 0$  such that, given any  $C^1$  disk  $D$  tangent to the center-unstable cone field,  $x \in D$  and  $n \geq 1$  an  $a/2$ -hyperbolic time for  $x$ , then*

$$d_{f^{n-k}(D)}(f^{n-k}(y), f^{n-k}(x)) \leq e^{-ka/2} d_{f^n(D)}(f^n(x), f^n(y)),$$

for any point  $y \in D$  with  $d_{f^n(D)}(f^n(x), f^n(y)) \leq \delta_1$ .

**Remark 6.5.** For fixed  $a > 0$ , we can take  $\delta_1$  to be constant for the diffeomorphisms in a  $C^1$  neighborhood of  $f$ .

**6.1.2. Physical measures with uniform size of basins.** Suppose  $f$  is a non-uniformly expanding along the center-unstable direction. By [5, Lemma 4.5],  $f$  admits an ergodic probability measure  $\mu$  such that the conditional measures of  $\mu$  along a family of local unstable manifolds are absolutely continuous with respect to Lebesgue measure. Moreover, by [5, Proposition 4.1], the size of this unstable manifolds are larger than  $\delta_1/4$ , where  $\delta_1$  is obtained by Lemma 6.4.

Then the basin of  $\mu$  contains a full Lebesgue measure subset of a local unstable manifold. Because the basin is saturated by stable leaves, and the stable foliation is absolute continuous,  $\mu$  is indeed a physical measure, whose basin contains Lebesgue almost every point of a ball with radius  $\delta_1/4$ .

And by [5, Corollary 4.6],  $f$  admits finitely many such physical measures, and the union of their basins has full volume. Combining with Remark 6.5, we have that:

**Proposition 6.6.** *Suppose  $f \in \text{Diff}^{1+}(M)$  is a non-uniformly expanding along the center-unstable direction, and  $\delta_1$  is obtained by Lemma 6.4. Then there is a  $C^1$  neighborhood  $\mathcal{U}$  of  $f$ , such that for any  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$ , if  $g$  is also a non-uniformly expanding along the center-unstable direction, then it admits finite many physical measures. The basin of each physical measure contains Lebesgue almost every point of a ball with radius  $\delta_1/4$ , and the union of these basins has full volume.*

**6.2. Basins with uniform size .** The main result of this subsection is the following proposition.

**Proposition 6.7.** *Suppose  $f$  has mostly expanding center, then there is a  $C^1$  neighborhood  $\mathcal{U}$  of  $f$  and  $\delta > 0$ , such that every  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$  admits finitely many physical measures, and the union of the basins has full volume. Moreover, the basin of each physical measure contains Lebesgue almost every point of some ball with radius  $\delta$ .*

*Proof.* By Proposition 5.5 and Theorem A, there exists a  $C^1$  neighborhood  $\mathcal{U}$  of  $f$ ,  $a > 0$  and  $N \in \mathbb{N}$ , such that for any  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$  and any  $\mu \in \text{Gibb}^u(g)$ ,

$$(27) \quad \int \log \|Dg^{-N}|_{E^{cs}(x)}\| d\mu(x) < -a.$$

**Lemma 6.8.** *There is  $N_0 \in \mathbb{N}$  such that, for any  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$ ,  $g^{N_0}$  is a no-uniformly expanding along the center-unstable direction, that is, for Lebesgue almost every  $x \in M$ , we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log \|Dg^{-N_0}|_{E^{cu}(g^{jN_0}(x))}\| < -a < 0.$$

*Proof.* Consider  $\nu_0$  an ergodic decomposition of  $\mu$  respect to the map  $g^N$ , denote by  $\nu_j = (g^j)_*\nu$ , then  $\nu_j = \nu_{j+N}$  and  $\mu = \frac{1}{N} \sum_{j=0}^{N-1} \nu_j$ .

By (27), there is some  $0 \leq j \leq N-1$  such that

$$\int \log \|Dg^{-N}|_{E^{cs}(x)}\| d\nu_j(x) < -a.$$

For simplicity, we assume that  $j = 0$ . Then for every point  $x$  in the basin of  $\mu$ , there is  $0 \leq j_0 \leq N-1$ , such that

$$\lim_n \frac{1}{n} \sum_{t=0}^{n-1} (g^{tN})_* \delta_{x_{j_0}} = \nu_0,$$

where  $x_{j_0} = g^{j_0}(x)$ . This implies that,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log \|Dg^{-N}|_{E^{cu}(g^{jN}(x_{j_0}))}\| < -a.$$

Replacing  $N$  by  $kN$  for  $k \in \mathbb{N}$ , we have

$$(28) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log \|Dg^{-Nk}|_{E^{cu}(g^{jNk}(x_{j_0}))}\| < -ka.$$

Hence, denoting  $C = \max \log \|Df\| + \max \log \|Df^{-1}\|$ ,

$$(29) \quad \begin{aligned} \log \|Dg^{-Nk}|_{E^{cu}(x_{j_0})}\| &= \log \|Dg^{-j_0} \circ Dg^{-Nk} \circ Dg^{j_0}|_{E^{cu}(x)}\| \\ &\geq \log \|Dg^{-Nk}|_{E^{cu}(x)}\| - 2Cj_0 \\ &\geq \log \|Dg^{-Nk}|_{E^{cu}(x)}\| - 2CN. \end{aligned}$$

Combining this inequality with (28), we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log \|Dg^{-Nk}|_{E^{cu}(g^{jNk}(x))}\| < -ka + 2CN < -a,$$

as long as we take  $k > \frac{2CN}{a} + 1$  and let  $N_0 = Nk$ . □

Let us continue the proof of Proposition 6.7.

By Lemma 6.8 and Proposition 6.6, there is  $\delta > 0$  such that for every  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$ ,  $g^{N_0}$  admits finite many physical measures  $\nu_{g,1}, \dots, \nu_{g,i(g)}$ , such that the basin of each physical measure contains Lebesgue almost every point of a ball with radius  $\delta$ , and the union of the basins covers a full volume subset. We conclude the proof of this proposition by the following observation: for each  $j = 1, \dots, i(g)$ ,

$$\mu = \frac{1}{N_0} \sum_{k=0}^{N_0-1} (g^k)_* \nu_{g,j}$$

is a physical measure of  $g$ , whose basin contains the basin of  $\nu_{g,j}$  for the map  $g^{N_0}$ . □

**6.3. Theorem C-part I:** Now we consider  $f$  to be a  $C^{1+}$  volume preserving partially hyperbolic diffeomorphism with one-dimensional center. By [15], the Lebesgue measure is ergodic, and by Birkhoff ergodic theorem, it is a physical measure and the basin has full volume. We further assume that

- (a) the center exponent of the Lebesgue measure is positive;
- (b)  $f$  is accessible.

We are going to show that there is a  $C^1$  neighborhood  $\mathcal{U}$  of  $f$ , such that every  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$  has mostly expanding center, with a unique physical measure, and the basin has full volume. The proof consists a series of lemmas.

**Lemma 6.9.**  *$f$  has mostly expanding center.*

*Proof.* We prove by contradiction. Suppose that  $f$  admits a Gibbs  $u$ -state  $\mu$  with non-positive center exponent, that is,

$$\int \lambda^c(x) d\mu(x) = \int \log \|df|_{E^c(x)}\| d\mu(x) \leq 0.$$

By (2) of Proposition 5.1, we may assume that  $\mu$  is in fact ergodic.

Since the Lebesgue measure has positive center exponent,  $\mu \neq \text{vol}$ , where  $\text{vol}$  denotes the Lebesgue measure. We claim that the center exponent of  $\mu$  cannot be negative: otherwise by (4) of Proposition 5.1,  $\mu$  is a physical measure, this contradicts that the basin of the Lebesgue measure has full volume.

It remains to show that the center exponent of  $\mu$  cannot be vanishing. Denote by  $\Lambda^u$  and  $\Lambda^s$  the sum of the Lyapunov exponents of measure  $\mu$  on the bundles  $E^u$  and  $E^s$  respectively, then

$$\Lambda^u = \int \log \det Df|_{E^u(x)} d\mu(x) \text{ and } \Lambda^s = \int \log \det Df|_{E^s(x)} d\mu(x).$$

Because  $f$  is volume preserving,

$$(30) \quad \Lambda^u + \Lambda^s = 0.$$

Then by Propositions 5.2 and 5.3,  $\mu$  satisfies the entropy formula, that is,

$$h_\mu(f) = h_\mu(f, \mathcal{F}^u) = \Lambda^u.$$

Because the metric entropy for  $f$  and  $f^{-1}$  coincide,  $h_\mu(f) = h_\mu(f^{-1})$ . By Proposition 5.3,  $\mu$  also satisfies the entropy formula for  $f^{-1}$ :

$$h_\mu(f^{-1}, \mathcal{F}^s) = h_\mu(f^{-1}) = h_\mu(f) = -\Lambda^s,$$

Applying Proposition 5.2 on  $f^{-1}$ , the above equality implies that  $\mu$  is a Gibbs  $s$ -state. By (1) of Proposition 5.1, the support of  $\mu$  is  $\mathcal{F}^u$  and  $\mathcal{F}^s$  saturated. Since  $f$  is accessible,  $\text{supp}(\mu)$  coincides to the whole manifold  $M$ .

Recall that both  $\text{vol}$  and  $\mu$  are Gibbs  $u$ -states, the conditional measure along the unstable leaves inside each foliation box are equivalent to the Lebesgue measure on the corresponding leaves.

We first take a  $\text{vol}$ -typical unstable plaque  $D$ , which means that, Lebesgue almost every point of  $D$  is the regular point of  $\text{vol}$ . In particular, they admit Pesin stable manifolds with dimension  $i_s + 1$ , where  $i_s = \dim(E^s)$ , and these Pesin stable manifolds are absolutely continuous and their union, denoted by  $\Gamma^s$  also belongs to the basin of  $\mu$ .

Because  $\text{supp}(\mu) = M$ , we may take a  $\mu$ -typical unstable disk  $D'$  which is sufficiently close to  $D$  such that Lebesgue almost every point of  $D'$  belongs to the basin of  $\mu'$ . But  $\Gamma^s$  intersects  $D'$  with a positive Lebesgue measure subset and the intersection belongs to the basin of  $\text{vol}$ , a contradiction.  $\square$



**Lemma 6.10.** *There is a  $C^1$  neighborhood of  $f$  such that, every  $C^{1+}$  diffeomorphism belongs to this neighborhood admits a unique physical measure, whose basin has full volume.*

*Proof.* By Lemma 6.9 and Proposition 6.7, there is  $\mathcal{U}$  a  $C^1$  neighborhood of  $f$  and  $\delta > 0$  such that every  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$  satisfies the following properties:

- has mostly expanding center,
- has finitely many physical measures, the union of the basins of physical measures have full volume;
- each basin contains Lebesgue almost every point of a ball with radius  $\delta$ .

Because  $f$  is transitive (by the ergodicity), we may take an open neighborhood  $\mathcal{U}_0 \subset \mathcal{U}$  of  $f$  such that, every diffeomorphism contained in  $\mathcal{U}_0$  is  $\delta$  transitive, that is, the positive iteration of every ball with radius  $\delta$  intersects any other ball with radius  $\delta$ .

Now we claim that every  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}_0$  admits a unique physical measure, since otherwise, the iteration of the basin of one physical measure will intersect the basin of a distinct physical measure. Note that the basin is invariant under iteration, then the basins of the two physical measures have non-trivial intersection, a contradiction.

The proof of the first part of Theorem C is finished.  $\square$

## 7. PHYSICAL MEASURES OF DIFFEOMORPHISMS WITH MOSTLY CONTRACTING CENTER

In this section, we prove the second part of Theorem C (see Subsection 7.3). In Subsection 7.1 we recall the main result of [16], where a relation between saddles and physical measures was built for the diffeomorphisms with mostly contracting center. And in Subsection 7.2, we show that the number of physical measures varies upper semi-continuously in the  $C^1$  topology.

Throughout this section, let  $f : M \rightarrow M$  be a  $C^{1+}$  partially hyperbolic diffeomorphism.

**7.1. diffeomorphisms with mostly contracting center.** Let  $f : M \rightarrow M$  has mostly contracting center. It was shown in [10] that:

**Proposition 7.1.**  *$f$  admits finitely many physical measures, such that the union of the basins has full volume. Moreover, there is a bijective map between physical measures and the ergodic Gibbs  $u$ -states of  $f$ .*

In the following, we give more precise description of the number of physical measures. We say that a hyperbolic saddle point has maximum index if the dimension of its stable manifold coincides with the dimension of the center stable bundle  $E^{cs}$ . A *skeleton* of  $f$  is a collection  $\mathcal{S} = \{p_1, \dots, p_k\}$  of hyperbolic saddle points with maximum index satisfying

- (i) For any  $x \in M$  there is  $p_i \in \mathcal{S}$  such that the stable manifold  $W^s(\text{Orb}(p_i))$  has some point of transversal intersection with the unstable leaf  $\mathcal{F}^u(x)$  through  $x$ ;
- (ii)  $W^s(\text{Orb}(p_i)) \cap W^u(\text{Orb}(p_j)) = \emptyset$  for every  $i \neq j$ , that is, the points in  $\mathcal{S}$  have no heteroclinic intersections.

The relation between a skeleton and physical measures for  $f$  was established in [16, Theorem A]:

**Proposition 7.2.** *Let  $f$  be a  $C^{1+}$  diffeomorphism with mostly contracting center. Then  $f$  admits some skeleton. Moreover, for any skeleton  $\mathcal{S} = \{p_1, \dots, p_k\}$  of  $f$ , the number of physical measures is precisely  $k = \#\mathcal{S}$ .*

Let us call *pre-skeleton* any finite collection  $\{p_1, \dots, p_k\}$  of saddles with maximum index satisfying condition (i), that is, such that every unstable leaf  $\mathcal{F}^u(x)$  has some point of transverse intersection with  $W^s(\text{Orb}(p_i))$  for some  $i$ . Thus a pre-skeleton is a skeleton if and only if there are no heteroclinic intersections between any of its points. More precisely,

**Lemma 7.3.** *[16, Lemma 2.5]*

*Every pre-skeleton contains a subset which is a skeleton.*

This notion of pre-skeleton is useful, since the continuation of a pre-skeleton is always a pre-skeleton:

**Lemma 7.4.** *[16, Lemma 2.4]* *Suppose  $f$  has a pre-skeleton  $\mathcal{S} = \{p_1, \dots, p_k\}$ . Let  $p_i(g), i = 1, \dots, k$  be the continuation of the saddles  $p_i$  for nearby diffeomorphism  $g$ . Then  $\mathcal{S}(g) = \{p_1(g), \dots, p_k(g)\}$  is a pre-skeleton for every  $g$  in a neighborhood of  $f$ .*

**7.2. Upper semi-continuation of number of physical measures.** In this subsection we generalize the main result of [1] into  $C^1$  topology:

**Proposition 7.5.** *Let  $f$  be a  $C^{1+}$  partially hyperbolic diffeomorphism with mostly contracting center. Suppose  $f$  has  $k$  number of physical measures, then there is a  $C^1$  neighborhood  $\mathcal{U}$  of  $f$ , such that for every  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$ ,  $g$  has mostly contracting center, and the number of physical measures of  $g$  is less than or equal to  $k$ .*

*Proof.* By Proposition 7.2,  $f$  admits a skeleton  $\mathcal{S} = \{p_1, \dots, p_k\}$  with  $k$  elements. Moreover, by Theorem B and Lemma 7.4, there is a  $C^1$  neighborhood  $\mathcal{U}$  of  $f$ , such that for every  $C^{1+}$  diffeomorphism  $g \in \mathcal{U}$ ,  $g$  has mostly contracting center, and  $\mathcal{S}(g) = \{p_1(g), \dots, p_k(g)\}$  is a pre-skeleton of  $g$ , where  $p_i(g), i = 1, \dots, k$  is the continuation of the saddle  $p_i$ .

Then by Lemma 7.3,  $\mathcal{S}(g)$  contains a subset which is a skeleton of  $g$ . As a corollary,  $g$  admits a skeleton with number of elements less than or equal to  $k$ . Then combining Proposition 7.2 and the fact that  $g$  has mostly contracting center, we show that the number of physical measures of  $g$  cannot exceed  $k$ . The proof is complete.  $\square$

**7.3. Theorem C-part II:.** Now we consider  $f$  to be a  $C^{1+}$  volume preserving partially hyperbolic diffeomorphism with one-dimensional center, such that

- (a) the integration center exponent is negative;
- (b) the unstable foliation  $\mathcal{F}^u$  is minimal, i.e., for every  $x \in M$ ,  $\mathcal{F}(x)$  is dense in the ambient manifold.

We are going to show that  $f$  admits a  $C^1$  open neighborhood, such that every  $C^{1+}$  diffeomorphism in this neighborhood has mostly contracting center, and it admits a unique physical measure whose basin has full volume.

By Proposition 7.5, we only need to show that such  $f$  has mostly contracting center and admits a unique physical measure.

Because the Lebesgue measure  $\text{vol}$  is a Gibbs  $u$ -state, by (2) of Proposition 5.1, almost every ergodic components of measure  $\text{vol}$  are Gibbs  $u$ -states. Then from the assumption that the integration of the center exponent of  $\text{vol}$  is negative:

$$\lambda^c(\text{vol}) = \int \log \|Df|_{E^c}\| d\text{vol} < 0,$$

there is an ergodic Gibbs  $u$ -state  $\mu$  with negative center exponent:

$$\lambda^c(\mu) = \int \log \|Df|_{E^c}\| d\mu < 0.$$

It remains to show that  $\mu$  is the unique Gibbs  $u$ -state of  $f$ . Suppose by contradiction that  $f$  contains another ergodic Gibbs  $u$ -state  $\mu_1$  of  $f$ . Because  $\mu$  is an ergodic Gibbs  $u$ -state with negative center exponent, there is a  $\mu$ -typical unstable disk  $D$  such that Lebesgue almost every point of  $D$  belongs to the basin of  $\mu$ , they admit Pesin stable manifolds with dimension  $i_s + 1$  which also belong to the basin, and these Pesin stable manifolds are absolutely continuous. We denote the union of these Pesin stable manifolds by  $\Gamma^s$ .

Note that by Proposition 5.1 (2),  $\text{supp}(\mu_1)$  is  $\mathcal{F}^u$  saturated, and because  $\mathcal{F}^u$  is minimal,  $\text{supp}(\mu_1) = M$ . We may take a  $\mu_1$ -typical unstable disk  $D_1$  which is arbitrarily close to  $D$ , such that Lebesgue almost every point of  $D_1$  belongs to the basin of  $\mu_1$ . Because by the absolute continuation of the Pesin stable manifolds,  $\Gamma^s$  intersects  $D_1$  with a Lebesgue positive measure subset inside  $D_1$ , which means that the basins of  $\mu$  and  $\mu_1$  have non-trivial intersection, a contradiction. The proof is complete.

#### APPENDIX A. PROOF OF PROPOSITION 3.1

*Proof of Proposition 3.1:* We need the following lemma which is modified from [20, Proposition 3.2].

**Lemma A.1.** *Let  $\mu$  be a probability measure supported on  $[0, R]$ . Then for any  $0 < \lambda < \lambda' < 1$ , there is a full Lebesgue subset  $I^* \subset (0, R)$  such that for every  $r \in I^*$ , there is  $D_r > 0$  satisfying that for every  $n \geq 0$ :*

$$\mu([r - \lambda^n, r + \lambda^n]) \leq (D_r \lambda')^n.$$

*Proof of Lemma A.1:* For each  $n$ , denote by

$$J_n = \{r \in [0, R]; \mu([r - \lambda^n, r + \lambda^n]) \geq (\lambda')^n\}.$$

Now consider a covering of  $J_n$  by  $\{(r_i - \lambda^n, r_i + \lambda^n); r_i \in J_n\}_{i=1, \dots, t}$  such that every point of interval is covered by at most twice. By the definition of  $J_n$ , we have

$$2 \geq \sum_{i=1}^t \mu((r_i - \lambda^n, r_i + \lambda^n)) \geq t(\lambda')^n.$$

Hence,  $t \leq \frac{2}{(\lambda')^n}$ . Then

$$\text{vol}(J_n) \leq \sum_{i=1}^t \text{vol}((r_i - \lambda^n, r_i + \lambda^n)) = t2\lambda^n \leq 4 \frac{\lambda^n}{(\lambda')^n},$$

which implies that  $\sum \text{vol}(J_n) < \infty$ .

Then for Lebesgue almost every  $r \in (0, R)$ , there is  $n_r$  such that  $r \notin J_n$  for any  $n \geq n_r$ . We can choose  $D_r > 1$  such that  $\mu((r - \lambda^i, r + \lambda^i)) < (D_r \lambda')^i$  for  $i = 1, \dots, n_r$ . The proof is complete.  $\square$

Let us continue the proof. Denote  $\mu_0 = \mu$  and  $K = \sum \frac{1}{n^2}$ , we consider a new probability measure  $\nu = \frac{1}{K}(\sum \frac{1}{(n+1)^2} \nu_n)$  of  $M$ . For every  $x \in M$ , denote the Borel measure  $\mu_x$  on  $[0, R]$  by

$$\mu_x((a, b)) = \nu(\{y : a \leq d(x, y) \leq b\}) \text{ where } (a, b) \subset [0, R].$$

If  $\mu_x([0, R]) = 0$ , we take  $r_x = \frac{R}{2}$ . Otherwise, applying Lemma A.1, we may choose  $\frac{R}{2} < r_x < R$  and  $D_x$  such that

$$\mu_x(\{y : r_x - \lambda^n \leq d(x, y) < r_x + \lambda^n\}) \leq D_x (\lambda')^n,$$

which is equivalent to say that:

$$(31) \quad \nu(\{y : r_x - \lambda^n \leq d(x, y) < r_x + \lambda^n\}) \leq D_x (\lambda')^n.$$

We can take finitely many points  $\{x_1, \dots, x_t\}$  such that  $B_{r_{x_i}}(x_i)$  covers  $M$ , and denote by  $D = \max_{1 \leq i \leq t} D_{x_i}$ . Then for the partition  $\mathcal{A} = \bigvee_{i=1}^t \{B_{r_{x_i}}(x_i), (B_{r_{x_i}}(x_i))^c\}$  and any  $i \geq 0$ , we have  $\text{diam}(\mathcal{A}) < R$  and

$$\nu(B_{\lambda^i}(\partial\mathcal{A})) \leq \sum_{j=1}^t \nu(B_{\lambda^i}(\partial(B_{r_{x_j}}(x_j)))) \leq tD(\lambda')^i.$$

Hence, for each  $n \geq 0$ ,  $\nu_n(B_{\lambda^i}(\partial\mathcal{A})) \leq K(1+n)^2 \nu(B_{\lambda^i}(\partial\mathcal{A})) \leq K(1+n)^2 tD(\lambda')^i$ . We conclude the proof by taking  $C_n = K(1+n)^2 tD$ .  $\square$

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DEPARTAMENTO DE GEOMETRIA, INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE  
FEDERAL FLUMINENSE, NITERÓI, BRAZIL  
*E-mail address:* yangjg@impa.br